

ZIGZAG STRUCTURE OF THIN CHAMBER COMPLEXES

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ABSTRACT. Zigzags in thin chamber complexes are investigated, in particular, all zigzags in the Coxeter complexes are described. Using this description, we show that the lengths of all zigzags in the simplex α_n , the cross-polytope β_n , the 24-cell, the icosahedron and the 600-cell are equal to the Coxeter numbers of A_n , $B_n = C_n$, F_4 and H_i , $i = 3, 4$, respectively. We also discuss in which cases two faces in a thin chamber complex can be connected by a zigzag.

1. INTRODUCTION

In the present paper, we investigate zigzags in *thin chamber complexes* which are interesting in the context of Tits buildings [15]. In fact, these are abstract polytopes, where all facets are simplices. The well-known Coxeter complexes form an important subclass of thin chamber complexes. The zigzags in a complex are the orbits of the action of a special operator T on the set of flags of this complex. The operator T transfers every flag F to a flag whose faces are adjacent to the faces of F . The main question under discussion is the following: in which cases can two faces of a complex be connected by a zigzag?

The notion of *Petrie polygon* for polytopes is one of the central concepts of famous Coxeter's book [2]. For embedded graph the same objects are appeared as *zigzags* in [3, 4, 6], *geodesics* in [8] and *left-right paths* in [14]. Their high-dimensional analogues are considered in [5] and [16], see also [6, Chapter 8]. Following [3, 4, 5, 6] we call such objects *zigzags*.

Consider an abstract polytope \mathcal{P} of rank n and one of its flags F . Let X_i be the i -face from this flag. There is the unique i -face X'_i adjacent to X_i and incident to all other faces from \mathcal{P} . We define $\sigma_i(F)$ as the flag obtained from F by replacing X_i on X'_i and introduce the operator

$$T = \sigma_{n-1} \dots \sigma_0$$

which acts on the set of all flags. In [5, 6] zigzags are defined as the orbits of this action. Similarly, for every permutation δ on the set $\{0, 1, \dots, n-1\}$ we consider the operator

$$T_\delta = \sigma_{\delta(n-1)} \dots \sigma_{\delta(0)}$$

and come to *generalized zigzags*. Note that such objects were first considered in [16] and named *Petrie schemes*.

Now, let us consider the associated flag complex $\mathfrak{F}(\mathcal{P})$ which is a thin chamber complex of the same rank n . In Subsection 3.4, we show that there is a natural one-to-one correspondence between generalized zigzags in \mathcal{P} and zigzags in $\mathfrak{F}(\mathcal{P})$. For this reason, it is natural to restrict the general zigzag theory on the case of zigzags in thin chamber complexes and associated with the operator T only.

Key words and phrases. zigzag, thin chamber complex, Coxeter complex.

We describe all zigzags of Coxeter complexes in terms of Coxeter elements and show that the length of zigzags depends on the associated Coxeter number (Subsection 3.3). For example, the Coxeter complexes for A_n , $B_n = C_n$, F_4 and H_i , $i = 3, 4$ are the flag complexes of the simplex α_n , the cross-polytope β_n , the 24-cell, the icosahedron and the 600-cell (respectively) and all zigzags in these complexes are induced by generalized zigzags in the above mentioned polytopes. In particular, this implies that the lengths of zigzags in the polytopes are equal to the corresponding Coxeter numbers. On the other hand, the Coxeter systems D_n and E_i , $i = 6, 7, 8$ are related to the half-cube $\frac{1}{2}\gamma_n$ and the E-polytopes 2_{21} , 3_{21} , 4_{21} (respectively). However, the associated Coxeter numbers are different from the lengths of zigzags in the polytopes. We explain why this occurs.

We say that two faces are *z-connected* if there is a zigzag joined them. If a thin chamber complex is *z-simple*, i.e. every zigzag is not self-intersecting, then the *z-connectedness* of any pair of faces implies that the complex is a simplex. The main result of this part concerns the *z-connectedness* of facets (Theorem 1). We consider the graph consisting of all facets, whose edges are pairs intersecting in a ridge. We determine a class of path geodesics in this graph which can be extended to zigzags. For *z-simple* complexes, this gives the full description of path geodesics extendible to zigzags.

2. THIN SIMPLICIAL COMPLEXES

2.1. Definitions and examples. Let Δ be an *abstract simplicial complex* over a finite set V , i.e. Δ is formed by subsets of V such that every one-element subset belongs to Δ and for every $X \in \Delta$ all subsets of X belong to Δ . Elements of Δ are called *faces* and maximal faces are said to be *facets*. We say that $X \in \Delta$ is a *k-face* if $|X| = k + 1$. Recall that 0-faces and 1-faces are known as *vertices* and *edges*, respectively, and the empty set is the unique (-1) -face. We will always suppose that the simplicial complex Δ is *pure*, i.e. all facets are of the same cardinality n . Then the number n is the *rank* of the simplicial complex.

Our second assumption is that Δ is *thin*. This means that every ridge, i.e. $(n-2)$ -face, is contained in precisely two distinct facets (cf. [15, Section 1.3]). Two facets are said to be *adjacent* if their intersection is a ridge.

Let k be a natural number not greater than $n-2$. If a $(k-1)$ -face Y is contained in a $(k+1)$ -face Z , then there are precisely two k -faces X_1 and X_2 such that

$$Y \subset X_i \subset Z \text{ for } i = 1, 2$$

(the set $Z \setminus Y$ contains only two vertices and X_1, X_2 are the k -faces containing Y and one of these vertices). We say that two distinct k -faces X_1 and X_2 are *adjacent* if there exist a $(k-1)$ -face Y and a $(k+1)$ -face Z satisfying the above inclusion.

For every $k \in \{0, 1, \dots, n-1\}$ we denote by $\Gamma_k(\Delta)$ the graph whose vertex set consists of all k -faces and whose edges are pairs of adjacent faces. This graph is one of so-called *Wythoff kaleidoscopes* [2]. Following [15, Section 1.3] we say that Δ is a *chamber complex* if the graph $\Gamma_{n-1}(\Delta)$ is connected. This condition guarantees that $\Gamma_k(\Delta)$ is connected for every k . Indeed, for any two k -faces X, Y with $k < n-1$ we take facets X', Y' containing X, Y (respectively); using a path in $\Gamma_{n-1}(\Delta)$ connecting X' with Y' we construct a path of $\Gamma_k(\Delta)$ connecting X with Y .

Let us consider a connected simple graph. The *path distance* $d(v, w)$ between vertices v and w in this graph is the smallest number d such that there is a path of length d connecting these vertices (see, for example, [7, Section 15.1]). Every path of length $d(v, w)$ connecting v and w is called a *geodesic*. The path distance on $\Gamma_k(\Delta)$ will be considered in Section 4.

Example 1. The n -simplex α_n is the simplicial complex whose vertex set is the $(n + 1)$ -element set

$$[n + 1] = \{1, \dots, n + 1\}$$

and whose non-empty faces are proper subsets of $[n + 1]$. The *cross-polytope* β_n is the simplicial complex whose vertex set is the set

$$[n]_{\pm} = \{1, \dots, n, -1, \dots, -n\}$$

and whose faces are all subsets $X \subset [n]_{\pm}$ such that for every $i \in X$ we have $-i \notin X$. It is clear that α_n and β_n are thin chamber complexes of rank n . Every $\Gamma_k(\alpha_n)$ is the Johnson graph $J(n + 1, k + 1)$ and $\Gamma_{n-1}(\beta_n)$ is the n -dimensional cube graph.

Example 2. Let Δ_1 and Δ_2 be thin chamber complexes over sets V_1 and V_2 , respectively. The *join* $\Delta_1 * \Delta_2$ is the simplicial complex whose vertex set is the disjoint union $V_1 \sqcup V_2$ and whose faces are all subsets of type $X_1 \sqcup X_2$, where $X_i \in \Delta_i$. This is a thin chamber complex of rank $n_1 + n_2$, where n_i is the rank of Δ_i .

Example 3. Let \mathcal{P} be a partially ordered set presented as the disjoint union of subsets $\mathcal{P}_{-1}, \mathcal{P}_0, \dots, \mathcal{P}_{n-1}, \mathcal{P}_n$ such that for any $X \in \mathcal{P}_i$ and $Y \in \mathcal{P}_j$ satisfying $X < Y$ we have $i < j$. The elements of \mathcal{P}_k are called k -faces. There is the unique (-1) -face and the unique n -face which are the minimal and maximal elements, respectively. Also, we suppose that every *flag*, i.e. a maximal linearly ordered subset, contains precisely n elements distinct from the (-1) -face and n -face. Then \mathcal{P} is an *abstract polytope* of rank n if the following conditions hold:

- (P1) If $k \in \{0, 1, \dots, n - 1\}$, then for any $(k - 1)$ -face Y and $(k + 1)$ -face Z satisfying $Y < Z$ there are precisely two k -faces X_i , $i = 1, 2$ such that $Y < X_i < Z$.
- (P2) \mathcal{P} is strongly connected (see, for example, [12]).

Every thin chamber complex of rank n can be considered as an abstract n -polytope whose $(n - 1)$ -faces are $(n - 1)$ -simplices. If \mathcal{P} is an abstract n -polytope, then the associated *flag complex* $\mathfrak{F}(\mathcal{P})$ is the simplicial complex whose vertices are the faces of \mathcal{P} and whose facets are the flags. This is a thin simplicial complex of rank n and (P2) guarantees that $\mathfrak{F}(\mathcal{P})$ is a chamber complex.

A simplicial complex is k -neighborly if any k distinct vertices form a face. Then α_n can be characterized as the unique n -neighborly thin chamber complex of rank n . We will use the following fact which follows from a more general result [9, p.123].

Fact 1. *If a thin chamber complex of rank n is k -neighborly and $k > \lfloor n/2 \rfloor$, then it is the n -simplex α_n .*

Example 4. The join $\alpha_n * \alpha_m$ with $n \leq m$ is a thin chamber complex of rank $n + m$ which is n -neighborly and not $(n + 1)$ -neighborly.

2.2. Coxeter complexes. Let W be a finite group generated by a set S whose elements are involutions and denoted by s_1, \dots, s_n . For any distinct $i, j \in [n]$ we write m_{ij} for the order of the element $s_i s_j$. Then $m_{ij} = m_{ji} \geq 2$ and the condition $m_{ij} = 2$ is equivalent to the fact that s_i and s_j commute. We suppose that (W, S) is a *Coxeter system*, i.e W is the quotient of the free group over S by the normal subgroup generated by all elements of type $(s_i s_j)^{m_{ij}}$. The associated *diagram* $D(W, S)$ is the graph whose vertex set is S and s_i is connected with s_j by an edge of order $m_{ij} - 2$ (the vertices are disjoint if $m_{ij} = 2$). All finite Coxeter systems are known [11], in particular, every finite irreducible Coxeter system is one of the following:

$$A_n, B_n = C_n, D_n, F_4, E_i, i = 6, 7, 8, H_i, i = 3, 4, I_2(m).$$

For every subset $I = \{i_1, \dots, i_k\} \subset [n]$ we denote by W^I the subgroup generated by the set

$$S \setminus \{s_{i_1}, \dots, s_{i_k}\}.$$

In particular, for every $i \in [n]$ the subgroup W^i is generated by $S \setminus \{s_i\}$. The following properties are well-known (see, for example, [1, Section 2.4]):

- (C1) $W^I \cap W^J = W^{I \cup J}$ for any subsets $I, J \subset [n]$,
- (C2) if $v, w \in W$ and $I, J \subset [n]$ then we have $wW^I = vW^J$ only in the case when $I = J$ and $w^{-1}v \in W^I$.

Also, we will use the following obvious equality

$$(1) \quad s_i W^j = W^j \quad \text{if } i \neq j.$$

The *Coxeter complex* $\Sigma(W, S)$ is the simplicial complex whose vertices are subsets of type wW^i with $w \in W$ and $i \in [n]$. The vertices X_1, \dots, X_k form a face if there exists $w \in W$ such that

$$X_1 = wW^{i_1}, \dots, X_k = wW^{i_k}.$$

This face can be identified with the set

$$X_1 \cap \dots \cap X_k = wW^I, \quad \text{where } I = \{i_1, \dots, i_k\}.$$

Every facet is of type $\{wW^1, \dots, wW^n\}$ and identified with the element w . So, there is a natural one-to-one correspondence between facets of $\Sigma(W, S)$ and elements of the group W .

Example 5. The Coxeter complexes for $A_n, B_n = C_n, F_4$ and $H_i, i = 3, 4$ are the flag complexes of α_n, β_n , the 24-cell, the icosahedron and the 600-cell, respectively. Note that the 24-cell is not a simplicial complex.

For every $w \in W$ the left multiplication L_w sending vW^i to wvW^i is an automorphism of the complex $\Sigma(W, S)$. Also, automorphisms of the diagram $D(W, S)$ (if they exist) induce automorphisms of $\Sigma(W, S)$. The automorphism group of $\Sigma(W, S)$ is generated by the left multiplications and the automorphisms induced by automorphisms of the diagram (this statement easily follows from [1, Corollary 3.2.6]).

The graph $\Gamma_{n-1}(\Sigma(W, S))$ coincides with the *Cayley graph* $C(W, S)$ whose vertex set is W and elements $w, v \in W$ are adjacent vertices if $v = ws_i$ for a certain $s_i \in S$. Indeed, maximal faces $\{wW^1, \dots, wW^n\}$ and $\{vW^1, \dots, vW^n\}$ are different

precisely in one vertex if and only if there is a unique $i \in [n]$ such that $wW^i \neq vW^i$, in other words,

$$w^{-1}v \in \bigcap_{j \neq i} W^j = \langle s_i \rangle$$

and we get the required equality.

Example 6. For the dihedral Coxeter system $I_2(m)$ the Cayley graph is the $(2m)$ -cycle. The Cayley graph of A_n is the 1-skeleton of the *permutohedron* [17]; in the general case, the corresponding polytope is called a *W-permutohedron* [10]. See [1, Figures 3.3] for the Cayley graph of H_3 .

The *length* $l(w)$ of an element $w \in W$ is the smallest length of an expression for w consisting of elements from S . Such an expression is called *reduced* if its length is equal to $l(w)$. Note that elements in a reduced expression are not necessarily mutually distinct. The path distance between $v, w \in W$ in the Cayley graph $C(W, S)$ is equal to $l(v^{-1}w) = l(w^{-1}v)$.

3. ZIGZAGS IN THIN SIMPLICIAL COMPLEXES

In this section, we will always suppose that Δ is a thin simplicial complex of rank n over a finite set V . Sometimes, Δ is assumed to be a thin chamber complex.

3.1. Flags. Every flag of Δ can be obtained from a certain sequence of n vertices which form a facet. Indeed, if x_0, x_1, \dots, x_{n-1} is such a sequence, then the corresponding flag is

$$(2) \quad \{x_0\} \subset \{x_0, x_1\} \subset \dots \subset \{x_0, x_1, \dots, x_{n-1}\}.$$

Any reenumeration of these vertices gives another flag which contains the facet consisting of the vertices.

If F is the flag corresponding to a vertex sequence x_0, x_1, \dots, x_{n-1} , then the flag $R(F)$ obtained from the reverse sequence x_{n-1}, \dots, x_1, x_0 is called the *reverse* of F , in other words, the reverse of (2) is the flag

$$\{x_{n-1}\} \subset \{x_{n-1}, x_{n-2}\} \subset \dots \subset \{x_{n-1}, \dots, x_1\} \subset \{x_{n-1}, \dots, x_1, x_0\}.$$

This definition is equivalent to [6, Definition 8.2].

Now, let F be the flag formed by X_0, \dots, X_{n-1} , where every X_i is an i -face. Then for every $i \in \{0, \dots, n-1\}$ there is the unique i -face X'_i adjacent to X_i and incident to all other X_j . We denote by $\sigma_i(F)$ the flag obtained from F by replacing X_i on X'_i . For every flag F we define

$$T(F) := \sigma_{n-1} \dots \sigma_1 \sigma_0(F).$$

Suppose that F is the flag obtained from a vertex sequence x_0, x_1, \dots, x_{n-1} . There is the unique vertex $x_n \neq x_0$ such that x_1, \dots, x_n form a facet. An easy verification shows that $T(F)$ is the flag corresponding to the sequence x_1, \dots, x_n .

Lemma 1. *For every flag F we have $TRT(F) = R(F)$.*

Proof. As above, we suppose that x_0, x_1, \dots, x_{n-1} and x_1, \dots, x_n are the sequences corresponding to the flags F and $T(F)$, respectively. Then the sequence x_n, \dots, x_1 corresponds to the flag $RT(F)$. This implies that $TRT(F)$ is defined by the sequence x_{n-1}, \dots, x_1, x_0 and we get the claim. \square

Using similar arguments we can prove the following.

Lemma 2. *If A is an automorphism of Δ , then for every flag F we have $AT(F) = TA(F)$.*

3.2. Zigzags and their shadows. For every flag F the sequence

$$Z = \{T^i(F)\}_{i \in \mathbb{N}}$$

(we assume that 0 belongs to \mathbb{N}) is called a *zigzag*. Since our simplicial complex is finite, we have $T^l(F) = F$ for some $l > 0$. The smallest number $l > 0$ satisfying this condition is said to be the *length* of the zigzag. For any number i the sequence

$$T^i(F), T^{i+1}(F), \dots$$

is also a zigzag; it is obtained from Z by a cyclic permutation of the flags. All such zigzags will be identified with Z .

The *k-shadow* of a zigzag $\{F_i\}_{i \in \mathbb{N}}$ is the sequence $\{X_i\}_{i \in \mathbb{N}}$, where every X_i is the k -face from the flag F_i .

Proposition 1. *Every zigzag can be uniquely reconstructed from any of the shadows.*

Proof. Let $Z = \{F_i\}_{i \in \mathbb{N}}$ be a zigzag of length l and let $\{X_i\}_{i \in \mathbb{N}}$ be the k -shadow of Z . The required statement is a consequence of the following two observations. If $k > 0$ then the $(k-1)$ -shadow is the sequence

$$X_{l-1} \cap X_0, X_0 \cap X_1, X_1 \cap X_2, \dots$$

Similarly, if $k < n-1$ then the $(k+1)$ -shadow consists of all $X_i \cup X_{i+1}$. \square

Proposition 2. *If a sequence $Z = \{F_1, \dots, F_l\}$ is a zigzag of length l , then the same holds for the sequence*

$$R(Z) = \{R(F_l), R(F_{l-1}), \dots, R(F_1)\}.$$

Proof. If $Z = \{F_1, \dots, F_l\}$ is a zigzag, then for every $i \in [l]$ we have $F_i = T^{i-1}(F)$, where $F = F_1$. By Lemma 1,

$$TR(F_i) = TRT^{i-1}(F) = RT^{i-2}(F) = R(F_{i-1})$$

if $i \geq 2$ and

$$TR(F_1) = TRT^l(F) = RT^{l-1}(F) = R(F_l).$$

This means that $R(Z)$ is a zigzag of length l . \square

Following [6, Definition 8.2], we say that the zigzag $R(Z)$ is the *reverse* of Z .

Remark 1. Let $Z = \{F_1, \dots, F_l\}$ be a zigzag. Denote by F_i^k the k -face belonging to the flag F_i . For every k the sequence F_1^k, \dots, F_l^k is the k -shadow of Z . The $(n-1)$ -shadow of the reverse zigzag is

$$F_l^{n-1}, F_{l-1}^{n-1}, \dots, F_1^{n-1}.$$

Then, by Proposition 1, the $(n-2)$ -shadow of the reverse zigzag is

$$F_1^{n-2}, F_l^{n-2}, F_{l-1}^{n-2}, \dots, F_2^{n-2}.$$

Step by step, we establish that

$$F_i^{n-i-1}, F_{i-1}^{n-i-1}, \dots, F_1^{n-i-1}, F_l^{n-i-1}, \dots, F_{i+1}^{n-i-1}$$

is the $(n - i - 1)$ -shadow of the reverse zigzag for every i satisfying $1 \leq i \leq n - 1$. In particular, if x_1, \dots, x_l is the 0-shadow of Z , then

$$x_{n-1}, x_{n-2}, \dots, x_1, x_l, x_{l-1}, \dots, x_n$$

is the 0-shadow of $R(Z)$.

Consider the flag F defined by a vertex sequence x_0, x_1, \dots, x_{n-1} , i.e.

$$\{x_0\} \subset \{x_0, x_1\} \subset \dots \subset \{x_0, x_1, \dots, x_{n-1}\}.$$

There is the unique facet containing x_1, \dots, x_{n-1} and distinct from the facet of F . In this facet, we take the unique vertex x_n distinct from x_1, \dots, x_{n-1} . It was noted above that $T(F)$ is the flag

$$\{x_1\} \subset \{x_1, x_2\} \subset \dots \subset \{x_1, \dots, x_n\}.$$

We apply the same arguments to the latter flag and get a certain vertex x_{n+1} . Recurrently, we construct a sequence of vertices $\{x_i\}_{i \in \mathbb{N}}$ such that $T^i(F)$ is the flag

$$(3) \quad \{x_i\} \subset \{x_i, x_{i+1}\} \subset \dots \subset \{x_i, \dots, x_{i+n-1}\}.$$

The sequence $\{x_i\}_{i \in \mathbb{N}}$ is the 0-shadow of the zigzag $\{T^i(F)\}_{i \in \mathbb{N}}$. For every $i \in \mathbb{N}$ the following assertions are fulfilled:

(Z1) $x_i, x_{i+1}, \dots, x_{i+n-1}$ form a facet,

(Z2) $x_i \neq x_{n+i}$.

If l is the length of the zigzag, then $l > n$ and $x_{i+l} = x_i$ for all $i \in \mathbb{N}$. The equality $x_i = x_j$ is possible for some distinct $i, j \in \{0, 1, \dots, l-1\}$ only in the case when $|i - j| > n$. We say that the zigzag $\{T^i(F)\}_{i \in \mathbb{N}}$ is *simple* if x_0, x_1, \dots, x_{l-1} are mutually distinct. In this case, the k -shadow is formed by l mutually distinct k -faces for every k . The complex Δ is said to be *z-simple* if every zigzag is simple.

Proposition 3. *If $\{x_i\}_{i \in \mathbb{N}}$ is a sequence of vertices satisfying (Z1) and (Z2) for every i , then all flags of type (3) form a zigzag and $\{x_i\}_{i \in \mathbb{N}}$ is the 0-shadow of this zigzag.*

Proof. Easy verification. □

Suppose that a zigzag Z is the reverse of itself, i.e. $R(Z)$ can be obtained from Z by a cyclic permutation of the flags. Then the 0-shadow of Z is a sequence of type

$$x_0, x_1, \dots, x_{n-1}, x_n, \dots, x_m, \dots, x_m, \dots, x_n, x_{n-1}, \dots, x_1, x_0, \dots$$

and for a sufficiently large m the distance between two exemplars of x_m is not greater than n . This contradicts (Z1). So, every zigzag is not the reverse of itself. In what follows, *every zigzag will be identified with its reverse*.

We say that Δ is *z-uniform* if all zigzags are of the same length.

Lemma 3. *If Δ is z-uniform and the length of zigzags is equal to l , then there are precisely $n!N/2l$ zigzags, where N is the number of facets in Δ .*

Proof. By the definition, zigzags are the orbits of the action of the operator T on the set of flags. Thus, the sum of the lengths of all zigzags is equal to the number of flags. There are precisely $n!N$ distinct flags. Since every zigzag is identified with its reverse, we get precisely $n!N/2l$ zigzags. □

Example 7. There is a natural one-to-one correspondence between zigzags of α_n and permutation on the set $[n + 1]$. Every zigzag is of length $n + 1$ and Lemma 3 implies that the number of zigzag is equal to $\frac{n!}{2}$.

There is the following characterizations of the n -simplex in terms of the length of zigzags.

Proposition 4. *Suppose that Δ is a thin chamber complex. Then it contains a zigzag of length $n + 1$ if and only if it is the n -simplex α_n .*

Proof. By Example 7, every zigzag in α_n is of length $n + 1$. Conversely, suppose that Z is a zigzag of length $n + 1$. It follows from (Z1) and (Z2) that the 0-shadow of Z consists of $n + 1$ mutually distinct vertices. Denote by X the subset of V formed by these $n + 1$ vertices. By (Z1), every n -element subset of X is a facet of Δ . Since every $(n - 1)$ -element subset of X is the intersection of two such facets, the connectedness of $\Gamma_{n-1}(\Delta)$ guarantees that there are no other facets in Δ . \square

Example 8. The 0-shadow of every zigzag in β_n is a sequence of the following type

$$i_1, \dots, i_n, -i_1, \dots, -i_n,$$

where i_1, \dots, i_n form a facet. Thus, all zigzags are of length $2n$. Since β_n has precisely 2^n facets, Lemma 3 shows that there are precisely $2^{n-2}(n - 1)!$ zigzags.

The 0-shadows of zigzags considered in Examples 7 and 8 consist of all vertices of the complex. For the general case this fails.

By Lemma 2, every automorphism of Δ sends zigzags to zigzags. We say that Δ is *z-transitive* if for any two zigzags there is an automorphism of Δ transferring one of them to the other. The complexes α_n , β_n are *z-transitive*. Also, they are *z-simple*.

3.3. Zigzags in Coxeter complexes. Let (W, S) be, as in Subsection 2.2, a finite Coxeter system and let s_1, \dots, s_n be the elements of S . For every permutation δ on the set $[n]$ we consider the corresponding *Coxeter element*

$$s_\delta := s_{\delta(1)} \dots s_{\delta(n)}.$$

It is well-known that the order of this element does not depend on δ [11, Section 3.16]. This order is denoted by h and called the *Coxeter number*. All Coxeter numbers are known [11, p.80, Table 2]. Also, we denote by E_δ the flag in $\Sigma(W, S)$ obtained from the following sequence of vertices

$$W^{\delta(1)}, \dots, W^{\delta(n)}.$$

The facet in this flag is identified with the identity element e . The second facet containing $W^{\delta(2)}, \dots, W^{\delta(n)}$ is

$$\{s_{\delta(1)}W^{\delta(2)} = W^{\delta(2)}, \dots, s_{\delta(1)}W^{\delta(n)} = W^{\delta(n)}, s_{\delta(1)}W^{\delta(1)}\}.$$

This facet is identified with $s_{\delta(1)}$. So, the flag $T(E_\delta)$ is defined by the sequence

$$W^{\delta(2)}, \dots, W^{\delta(n)}, s_{\delta(1)}W^{\delta(1)}.$$

The facet containing $W^{\delta(3)}, \dots, W^{\delta(n)}, s_{\delta(1)}W^{\delta(1)}$ and distinct from $s_{\delta(1)}$ is

$$\{s_{\delta(1)}s_{\delta(2)}W^{\delta(3)} = W^{\delta(3)}, \dots, s_{\delta(1)}s_{\delta(2)}W^{\delta(1)} = s_{\delta(1)}W^{\delta(1)}, s_{\delta(1)}s_{\delta(2)}W^{\delta(2)}\}.$$

This facet is identified with $s_{\delta(1)}s_{\delta(2)}$ and the flag $T^2(E_\delta)$ is related to the vertex sequence

$$W^{\delta(3)}, \dots, W^{\delta(n)}, s_{\delta(1)}W^{\delta(1)}, s_{\delta(1)}s_{\delta(2)}W^{\delta(2)}.$$

Similarly, we show that for every $i \in [n-1]$ the flag $T^i(E_\delta)$ is defined by the sequence

$$W^{\delta(i+1)}, \dots, W^{\delta(n)}, s_{\delta(1)}W^{\delta(1)}, s_{\delta(1)}s_{\delta(2)}W^{\delta(2)}, \dots, s_{\delta(1)} \dots s_{\delta(i)}W^{\delta(i)}$$

and the flag $T^n(E_\delta)$ corresponds to the sequence

$$s_{\delta(1)}W^{\delta(1)}, s_{\delta(1)}s_{\delta(2)}W^{\delta(2)}, \dots, s_{\delta(1)} \dots s_{\delta(n)}W^{\delta(n)}.$$

Using (1) we rewrite the latter sequence as follows

$$s_\delta W^{\delta(1)}, \dots, s_\delta W^{\delta(n)}.$$

Therefore,

$$T^n(E_\delta) = L_{s_\delta}(E_\delta)$$

(recall that L_w is the left multiplication sending every vW^i to wvW^i , see Subsection 2.2). Since L_{s_δ} is an automorphism of $\Sigma(W, S)$, Lemma 2 implies that

$$T^{n+i}(E_\delta) = L_{s_\delta}T^i(E_\delta)$$

for every $i \in \mathbb{N}$, in particular,

$$T^{mn}(E_\delta) = L_{s_\delta^m}(E_\delta).$$

Thus the length of the zigzag $\{T^i(E_\delta)\}_{i \in \mathbb{N}}$ is equal to nh and the 0-shadow is

$$W^{\delta(1)}, \dots, W^{\delta(n)}, s_\delta W^{\delta(1)}, \dots, s_\delta W^{\delta(n)}, \dots, s_\delta^{h-1}W^{\delta(1)}, \dots, s_\delta^{h-1}W^{\delta(n)}.$$

The $(n-1)$ -shadow of this zigzag is the following

$$e, s_{\delta(1)}, s_{\delta(1)}s_{\delta(2)}, \dots, s_\delta, s_\delta s_{\delta(1)}, \dots, s_\delta^2, s_\delta^2 s_{\delta(1)}, \dots, s_\delta^h = e.$$

For every number $m < h$ the element s_δ^m does not belong to any W^i (see [16, Theorem 3.1]). This implies that the zigzag $\{T^i(E_\delta)\}_{i \in \mathbb{N}}$ is simple.

For every flag F in $\Sigma(W, S)$ there exist $w \in W$ and a permutation δ on the set $[n]$ such that $F = L_w(E_\delta)$. The automorphism L_w sends the zigzag $\{T^i(E_\delta)\}_{i \in \mathbb{N}}$ to the zigzag $\{T^i(F)\}_{i \in \mathbb{N}}$. The 0-shadow of the latter zigzag is

$$(4) \quad wW^{\delta(1)}, \dots, wW^{\delta(n)}, \dots, ws_\delta^{h-1}W^{\delta(1)}, \dots, ws_\delta^{h-1}W^{\delta(n)}$$

and the $(n-1)$ -shadow is

$$(5) \quad w, ws_{\delta(1)}, ws_{\delta(1)}s_{\delta(2)}, \dots, ws_\delta, ws_\delta s_{\delta(1)}, \dots, ws_\delta^2, ws_\delta^2 s_{\delta(1)}, \dots, ws_\delta^h = w.$$

Since $\Sigma(W, S)$ is z -uniform and contains precisely $|W|$ facets, we can find the number of zigzags in $\Sigma(W, S)$ using Lemma 3.

So, we get the following.

Proposition 5. *The following assertions are fulfilled:*

- (1) *the Coxeter complex $\Sigma(W, S)$ is z -simple;*
- (2) *there are precisely $|W|(n-1)!/2h$ distinct zigzags in $\Sigma(W, S)$ and the length of every zigzag is equal to nh , where h is the Coxeter number corresponding to (W, S) and $n = |S|$;*
- (3) *the 0-shadow and the $(n-1)$ -shadow of every zigzag in $\Sigma(W, S)$ are described by the formulas (4) and (5), respectively.*

The existence of an automorphism of $\Sigma(W, S)$ transferring the flag defined by the sequence W^1, \dots, W^n to the flag associated to the sequence $W^{\delta(1)}, \dots, W^{\delta(n)}$ is equivalent to the fact that the permutation δ induces an automorphism of the diagram $D(W, S)$. So, $\Sigma(W, S)$ is z -transitive only in some special cases when every permutation on the set $[n]$ induces an automorphism of the diagram.

3.4. Generalized zigzags. Let δ be a permutation on the set $\{0, 1, \dots, n-1\}$. Consider the operator

$$T_\delta := \sigma_{\delta(n-1)} \dots \sigma_{\delta(1)} \sigma_{\delta(0)}$$

on the set of flags in Δ . For every flag F the sequence $\{T_\delta^i(F)\}_{i \in \mathbb{N}}$ will be called a δ -zigzag. Also, we say that this is a *generalized zigzag*. As in Subsection 3.2, we define the *length* and *shadows* of generalized zigzags.

Remark 2. Let F be a flag in Δ and let $Z = \{F, T(F), \dots, T^{l-1}(F)\}$ be the associated zigzag. The operator $\tilde{T} = \sigma_0 \sigma_1 \dots \sigma_{n-1}$ coincides with T^{-1} and we have

$$\begin{aligned} \tilde{T}(F) &= T^{l-1}(F), \\ \tilde{T}^2(F) &= T^{2l-2}(F) = T^{l-2}(F), \\ &\vdots \\ \tilde{T}^{l-1}(F) &= T^{(l-1)^2}(F) = T(F). \end{aligned}$$

Therefore, $\{F, \tilde{T}(F), \dots, \tilde{T}^{l-1}(F)\}$ is the zigzag reversed to Z . Similarly, we show that the generalized zigzags defined by the operator $\sigma_{\delta(0)} \sigma_{\delta(1)} \dots \sigma_{\delta(n-1)}$ are reversed to the generalized zigzags obtained from T_δ .

Let F be a flag of Δ whose k -face is denoted by X_k for every $k \in \{0, 1, \dots, n-1\}$. This is a facet in the flag complex $\mathfrak{F}(\Delta)$ and we consider the zigzag Z in $\mathfrak{F}(\Delta)$ defined by the vertex sequence

$$X_{\delta(0)}, X_{\delta(1)}, \dots, X_{\delta(n-1)},$$

where δ is a certain permutation on the set $\{0, 1, \dots, n-1\}$. Let $\{Y_i\}_{i \in \mathbb{N}}$ be the 0-shadow of this zigzag. Then

$$Y_0 = X_{\delta(0)}, Y_1 = X_{\delta(1)}, \dots, Y_{n-1} = X_{\delta(n-1)}$$

and Y_n is the $\delta(0)$ -face in the flag $\sigma_{\delta(0)}(F)$. Similarly, Y_{n+1} is the $\delta(1)$ -face in the flag $\sigma_{\delta(1)} \sigma_{\delta(0)}(F)$. Step by step, we establish that Y_{n+i} is the $\delta(i)$ -face in the flag

$$\sigma_{\delta(i)} \dots \sigma_{\delta(1)} \sigma_{\delta(0)}(F)$$

for every $i \in \{0, 1, \dots, n-1\}$. Therefore, $Y_n, Y_{n+1}, \dots, Y_{2n-1}$ belongs to the flag $T_\delta(F)$. The same arguments show that Y_{kn+i} is the $\delta(i)$ -face in the flag $T_\delta^k(F)$ for every $k \in \mathbb{N}$ and $i \in \{0, 1, \dots, n-1\}$. In other words, the 0-shadow of Z is formed by the faces from the δ -zigzag $\{T_\delta^i(F)\}_{i \in \mathbb{N}}$, i.e.

$$\underbrace{Y_0, Y_1, \dots, Y_{n-1}}_F, \underbrace{Y_n, \dots, Y_{2n-1}}_{T_\delta(F)}, \underbrace{Y_{2n}, \dots, Y_{3n-1}}_{T_\delta^2(F)}, \dots$$

Thus the length of Z is equal to nl , where l is the length of $\{T_\delta^i(F)\}_{i \in \mathbb{N}}$. It is trivial that Z is simple if and only if $\{T_\delta^i(F)\}_{i \in \mathbb{N}}$ is simple. In particular, we have proved the following.

Proposition 6. *There is a natural one-to-one correspondence between zigzags in $\mathfrak{F}(\Delta)$ and generalized zigzags in Δ . The length of a zigzag in $\mathfrak{F}(\Delta)$ is equal to nl , where l is the length of the corresponding generalized zigzag in Δ .*

Lemma 3 implies the following.

Corollary 1. *If $\mathfrak{F}(\Delta)$ is z -uniform and the length of generalized zigzags in Δ is equal to l , then there are precisely $(n-1)!N/2l$ generalized zigzags, where N is the number of flags in Δ .*

Clearly, generalized zigzags can be defined in abstract polytopes and Proposition 6 holds for this case. Recall that the flag complexes of α_n , β_n , the 24-cell, the icosahedron and the 600-cell are the Coxeter complexes of A_n , $B_n = C_n$, F_4 and H_i , $i = 3, 4$ (respectively). Propositions 5 and 6 imply the following.

Corollary 2. *The lengths of generalized zigzags in α_n , β_n , the 24-cell, the icosahedron and the 600-cell are equal to the corresponding Coxeter numbers*

$$h(A_n) = n + 1, \quad h(B_n) = 2n, \quad h(F_4) = 12, \quad h(H_3) = 10, \quad h(H_4) = 30,$$

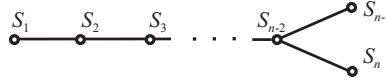
respectively.

Note that all values from Corollary 1 were given in [5, 6], but the connection with Coxeter numbers is new.

Now, we consider the half-cube $\frac{1}{2}\gamma_n$ and the E-polytopes 2_{21} , 3_{21} , 4_{21} associated to the Coxeter systems D_n and E_i , $i = 6, 7, 8$ (respectively). The zigzag lengths of these polytopes are not equal to the corresponding Coxeter numbers. The Coxeter numbers of E_i , $i = 6, 7, 8$, are 12, 18, 30 and, by [5, 6], the zigzag lengths of the E-polytopes are 18, 90, 36. The Coxeter number of D_n is equal to $2(n-1)$ and the zigzag length of $\frac{1}{2}\gamma_n$ can be found in [5, 6] for $n \leq 13$.

The following example explains this non-coincidence.

Example 9. For every $i \in [n-1]$ we denote by \mathcal{F}_i the set of all i -faces in β_n . Then \mathcal{F}_{n-1} can be presented as the disjoint union of two subsets \mathcal{F}_+ and \mathcal{F}_- satisfying the following condition: for any $X, Y \in \mathcal{F}_s$, $s \in \{+, -\}$ the number $n - |X \cap Y|$ is even and this number is odd if $X \in \mathcal{F}_+$ and $Y \in \mathcal{F}_-$. Suppose that (W, S) is the Coxeter system of type D_n (the corresponding diagram is on the figure below).



Recall that the vertices of the Coxeter complex $\Sigma(W, S)$ are all subsets wW^i , $w \in W$ and $i \in [n]$. For every $i \in [n-2]$ the vertices of type wW^i correspond to the elements of \mathcal{F}_{i-1} and the vertices of types wW^{n-1} and wW^n correspond to the elements of \mathcal{F}_+ and \mathcal{F}_- (respectively), see [13, Example 2.7] for the details. For every $i \in [n-1]$ the i -faces of the hypercube γ_n can be identified with the elements of \mathcal{F}_{n-1-i} . In the case when the vertex set of the half-cube $\frac{1}{2}\gamma_n$ is \mathcal{F}_+ , the 2-faces of $\frac{1}{2}\gamma_n$ are the element of $\mathcal{F}_{n-3} \cup \mathcal{F}_-$. In other words, the 2-faces of $\frac{1}{2}\gamma_n$ give two different types of vertices in the Coxeter complex. This means that zigzags in the Coxeter complex of D_n cannot be obtained from generalized zigzags in $\frac{1}{2}\gamma_n$ as it was described in Proposition 6. For the same reason zigzags in the Coxeter complexes of E_i , $i = 6, 7, 8$ are not induced by generalized zigzags of the associated E-polytopes.

4. Z-CONNECTEDNESS OF FACES

In this section, we suppose that Δ is a thin chamber complex of rank n . Then every $\Gamma_k(\Delta)$ is connected. We say that two faces X and Y in Δ are z -connected if there is a zigzag such that X and Y are contained in some flags of this zigzag.

Example 10. In the n -simplex α_n , any two faces are z -connected. This easily follows from the fact that every zigzag is defined by a certain enumeration of the set $[n + 1]$ (Example 7).

Example 11. If $n \geq 3$, then the complex β_n contains pairs of faces which are not z -connected. Consider, for example, the edges

$$\{i, j\} \text{ and } \{i, -j\}.$$

Up to a cyclic permutation, the 0-shadow of a zigzag containing $\{i, j\}$ is of type

$$i, j, i_1, \dots, i_{n-2}, -i, -j, -i_1, \dots, -i_{n-2},$$

where each of i_1, \dots, i_{n-2} is not equal to $\pm i$ or $\pm j$ (see Example 8). The corresponding zigzag does not contain $\{i, -j\}$. Similarly, we can show that the k -faces

$$\{i_1, \dots, i_k, j\} \text{ and } \{i_1, \dots, i_k, -j\}$$

are not z -connected if $k < n - 1$. Recall that $\Gamma_{n-1}(\beta_n)$ is the n -dimensional cube graph. It is easy to see that every geodesic of this graph is contained in the $(n - 1)$ -shadow of a certain zigzag.

4.1. Z -connectedness of facets and distance normal geodesics. If Δ is α_n or β_n , then

$$(6) \quad d(X, Y) = n - |X \cap Y|$$

for any two facets X, Y . Recall that $d(X, Y)$ is the path distance between X and Y in $\Gamma_{n-1}(\Delta)$. For the general case this distance formula fails. Consider, for example, an n -gonal bipyramid; if $n \geq 6$, then it contains two faces X, Y with a common vertex and such that $d(X, Y) > 2$. In the general case, we have

$$d(X, Y) \geq n - |X \cap Y|$$

for any two facets X, Y in Δ .

We say that facets X and Y satisfying $d(X, Y) \leq n$ form a *distance normal pair* if the equality (6) holds. Any two adjacent facets form a distance normal pair. If the path distance between two facets is equal to 2, then they form a distance normal pair. It was noted above that any two facets in α_n or β_n form a distance normal pair. Every geodesic joining distance normal pair X, Y will be called *distance normal geodesics* if $d(X, Y) \leq n$.

Example 12. Let (W, S) be a Coxeter system and $|S| = n$. Consider the associated Coxeter complex $\Sigma(W, S)$. By Subsection 2.2, the graph $\Gamma_{n-1}(\Sigma(W, S))$ can be naturally identified with the Cayley graph $C(W, S)$. Let s_i and s_j be non-commuting elements of S . The path distance between the identity element e and $w = s_i s_j s_i$ in $C(W, S)$ is equal to 3. On the other hand, e and w correspond to the facets $\{W^1, \dots, W^n\}$ and $\{wW^1, \dots, wW^n\}$, respectively. Since we have $wW^k = W^k$ for every $k \neq i, j$, the intersection of these facets contains precisely $n - 2$ vertices and they do not form a distance normal pair.

The latter example can be generalized as follows.

Example 13. Let w and v be elements of W such that $l(w^{-1}v) \leq n$. Then $d(w, v) \leq n$. The facets of $\Sigma(W, S)$ corresponding to w and v form a distance normal pair if and only if there is a reduced expression for $w^{-1}v$ whose elements are mutually distinct (the existence of such an expression implies that all reduced expressions of $w^{-1}v$ satisfy the same condition).

Now, we consider facets X and Y such that $d(X, Y) > n$. We say that X and Y form a *distance normal pair* if there exists a geodesic

$$X = X_0, X_1, \dots, X_m = Y,$$

where any two X_i, X_j satisfying $d(X_i, X_j) \leq n$ form a distance normal pair. Every such a geodesic will be called *distance normal*. The fact that two facets are connected by a distance normal geodesic does not guarantee that every geodesic connecting them is distance normal.

Example 14. It is not difficult to construct a thin chamber complex of rank 3 satisfying the following conditions:

- there is a distance normal pair of faces X, Y such that $d(X, Y) = 4$,
- there is a face X' adjacent to X and intersecting Y precisely in a vertex,
- $d(X', Y) = 3$.

So, X' and Y do not form a distance normal pair. Let X', X_1, X_2, Y be a geodesic connecting X' and Y . Then X, X', X_1, X_2, Y is a geodesic from X to Y which is not distance normal.

The $(n-1)$ -shadows of zigzags in Δ are closed (not necessarily simple) paths in $\Gamma_{n-1}(\Delta)$. We say that a path X_1, \dots, X_m is contained in a path Y_1, \dots, Y_k (or Y_1, \dots, Y_k contains X_1, \dots, X_m) if there is a number j such that $Y_{j+i} = X_i$ for every $i \in \{0, 1, \dots, m\}$. It is clear that every path contained in a distance normal geodesic is a distance normal geodesic.

Lemma 4. *If Z is a simple zigzag in Δ , then every geodesic of $\Gamma_{n-1}(\Delta)$ contained in the $(n-1)$ -shadow of Z is distance normal.*

Proof. Easy verification. □

Theorem 1. *Every distance normal geodesic of $\Gamma_{n-1}(\Delta)$ is contained in the $(n-1)$ -shadow of a certain zigzag of Δ and the following assertions are fulfilled:*

- (1) *if this geodesic is of length $m \leq n$, then there are at most $(n-m)!$ zigzags whose $(n-1)$ -shadows contain the geodesic;*
- (2) *if the length of the geodesic is greater than n , then it is contained in the $(n-1)$ -shadow of the unique zigzag.*

Theorem 1 together with Lemma 4 give the following.

Corollary 3. *Suppose that Δ is z -simple. A geodesic of $\Gamma_{n-1}(\Delta)$ is contained in the $(n-1)$ -shadow of a certain zigzag if and only if it is distance normal.*

Remark 3. Example 13 shows that for the Coxeter complexes Theorem 1 easily follows from Proposition 5.

4.2. Proof of Theorem 1. Let X_0, X_1, \dots, X_m be a distance normal geodesic in $\Gamma_{n-1}(\Delta)$. First, we consider the case when $m \leq n$ and prove the statement by induction.

If $m = 1$, then we take vertices x_0, x_1, \dots, x_n such that

$$X_0 = \{x_0, x_1, \dots, x_{n-1}\} \text{ and } X_1 = \{x_1, \dots, x_n\}.$$

For every permutation δ on the set $[n-1]$ we consider the sequence

$$x_0, x_{\delta(1)}, \dots, x_{\delta(n-1)}$$

and denote by F_δ the associated flag

$$\{x_0\} \subset \{x_0, x_{\delta(1)}\} \subset \cdots \subset \{x_0, x_{\delta(1)}, \dots, x_{\delta(n-1)}\} = X_0.$$

Since the vertices $x_{\delta(1)}, \dots, x_{\delta(n-1)}$ belong to X_1 , the facet of the flag $T(F_\delta)$ is X_1 . It is easy to see that every zigzag whose $(n-1)$ -shadow contains the path X_0, X_1 is of type $\{T^i(F_\delta)\}_{i \in \mathbb{N}}$. Since Δ is not assumed to be z -simple, the zigzags corresponding to the distinct flags F_δ and F_γ may be coincident or the zigzag defined by F_δ is the reverse of the zigzag defined by F_γ . Therefore, there are at most $(n-1)!$ zigzags satisfying the required condition.

Let $m > 1$. Then X_0, X_1, \dots, X_{m-1} is a distance normal geodesic and, by the inductive hypothesis, it is contained in the $(n-1)$ -shadow of a certain zigzag Z . Let $\{x_i\}_{i \in \mathbb{N}}$ be the 0-shadow of Z . We suppose that the vertices x_0, x_1, \dots, x_{n-1} belong to X_0 . Then

$$X_i = \{x_i, \dots, x_{i+n-1}\}$$

for every $i \in [m-1]$.

If $n = m$, then $X_0 \cap X_m = \emptyset$, in particular, $x_{n-1} \notin X_n$. Since

$$X_{n-1} = \{x_{n-1}, \dots, x_{2n-2}\}$$

and X_n are adjacent, the vertices x_n, \dots, x_{2n-2} belong to X_n . This implies that X_n coincides with the facet $\{x_n, \dots, x_{2n-1}\}$ and the geodesic X_0, X_1, \dots, X_n is contained in the $(n-1)$ -shadow of Z . This is the unique zigzag whose $(n-1)$ -shadow contains this geodesic. Indeed, every x_i , $i \in \{0, 1, \dots, n-1\}$ is the unique vertex belonging to $X_i \setminus X_{i+1}$ and Z is completely determined by the list of the first n vertices in the 0-shadow.

Consider the case when $m < n$. The face

$$A = \{x_{m-1}, \dots, x_{n-1}\}$$

consists of $n - m + 1 \geq 2$ vertices and coincides with the intersection of X_0 and X_{m-1} . Since X_0 and X_m form a distance normal pair and $d(X_0, X_m) = m$, the face $A \cap X_m$ contains precisely $n - m$ vertices. So, there is the unique number $t \in \{m-1, \dots, n-1\}$ such that $x_t \notin X_m$. For every permutation δ on the set $\{m-1, \dots, n-1\} \setminus t$ we consider the vertex sequence

$$(7) \quad x_0, x_1, \dots, x_{m-2}, x_t, x_{\delta(m-1)}, \dots, \hat{x}_t, \dots, x_{\delta(n-1)}$$

(the symbol $\hat{}$ means that the corresponding element is omitted) and denote by F_δ the flag obtained from this sequence. This flag defines the zigzag

$$Z_\delta = \{T^i(F_\delta)\}_{i \in \mathbb{N}}.$$

Obviously, the first n vertices in the 0-shadow of Z_δ are (7). The next $m-1$ vertices are x_n, \dots, x_{n+m-2} which means that the $(n-1)$ -shadow of Z_δ contains the geodesic X_0, X_1, \dots, X_{m-1} . The m -th element in the $(n-1)$ -shadow of Z_δ is adjacent to X_{m-1} and does not contain x_t . This implies that it coincides with X_m . Therefore, the geodesic X_0, X_1, \dots, X_m is contained in the $(n-1)$ -shadow of Z_δ . Using the fact that x_i is the unique vertex in $X_i \setminus X_{i+1}$ for $i \in \{0, 1, \dots, m-2\}$, x_t is the unique vertex in $X_{m-1} \setminus X_m$ and

$$X_0 \cap X_m = X_0 \cap X_1 \cap \cdots \cap X_m = A \setminus \{x_t\},$$

we show that every zigzag whose $(n-1)$ -shadow contains X_0, X_1, \dots, X_m is of type Z_δ . As above, for distinct permutations δ and γ the zigzags Z_δ and Z_γ may be coincident. For this reason, there are at most $(n-m)!$ such zigzags.

Now, we suppose that $m > n$. For every $i \in \{0, 1, \dots, m - n\}$ we have

$$d(X_i, X_{i+n}) = n$$

and the facets X_i, X_{i+n} form a distance normal pair. It was established above that there is the unique zigzag Z_i whose $(n - 1)$ -shadow contains the geodesic X_i, \dots, X_{i+n} . If $i < m - n$, then the geodesic X_{i+1}, \dots, X_{i+n} is contained in the $(n - 1)$ -shadows of Z_i and Z_{i+1} . Since

$$d(X_{i+1}, X_{i+n}) = n - 1,$$

there is the unique zigzag whose $(n - 1)$ -shadow contains X_{i+1}, \dots, X_{i+n} . Thus Z_i coincides with Z_{i+1} for every $i < n - m$. This means that all zigzags Z_i are coincident and we get the claim.

4.3. Z -connectedness of non-maximal faces. Two faces X and Y of the same non-maximal rank $k \geq 1$ are said to be *weakly adjacent* if their intersection is a $(k - 1)$ -face and there is no face containing both X and Y . This is a modification of the relation defined for non-maximal singular subspaces of polar spaces [13, Subsection 4.6.2]. Note that the non-maximal faces of β_n considered in Example 11 are weakly adjacent.

Lemma 5. *For every $k \in [n - 2]$ any pair of weakly adjacent k -faces in Δ cannot be connected by a simple zigzag.*

Proof. Let $\{x_i\}_{i \in \mathbb{N}}$ be the 0-shadow of a zigzag connecting weakly adjacent k -faces X and Y . If the zigzag is simple, then there exists i such that

$$X \cap Y = \{x_{i+1}, \dots, x_{i+k}\},$$

one of X, Y is $\{x_i, \dots, x_{i+k}\}$ and the other is $\{x_{i+1}, \dots, x_{i+k+1}\}$. Since the rank k is not maximal, we have $k + 2 \leq n$ and

$$X \cup Y = \{x_i, \dots, x_{i+k+1}\}$$

is a face which contradicts the fact that X and Y are weakly adjacent. \square

A possible z -connectedness for two weakly adjacent faces is described in the following example.

Example 15. Let $\{x_i\}_{i \in \mathbb{N}}$ be the 0-shadow of a zigzag. The length l of the zigzag is assumed to be sufficiently large. Also, we suppose that the zigzag is not simple and there exist $i, j \in [l - 1]$ and $k < n - 1$ such that

$$i + k \leq j, \quad j + k \leq l - 1$$

and

$$x_i = x_j, x_{i+1} = x_{j+1}, \dots, x_{i+k-1} = x_{j+k-1}.$$

If the vertices $x_{i-1}, x_i, \dots, x_{i+k-1}, x_{j+k}$ do not form a face (this means that $j + k - i > n - 1$), then

$$\{x_{i-1}, x_i, \dots, x_{i+k-1}\} \quad \text{and} \quad \{x_j, \dots, x_{j+k-1}, x_{j+k}\}$$

are weakly adjacent k -faces connected by our zigzag.

Lemma 6. *If Δ is z -simple and any two edges of Δ are z -connected, then Δ is 3-neighborly.*

Proof. By Lemma 5, there exist no pairs of weakly adjacent edges, i.e. any two edges with a common vertex are adjacent. If x_0, x_1, \dots, x_m is a path in $\Gamma_0(\Delta)$, then the edges x_0x_1 and x_1x_2 are adjacent which implies that x_0, x_2 are adjacent vertices and x_0, x_2, \dots, x_m is a path in $\Gamma_0(\Delta)$. Step by step, we show that the vertices x_0 and x_m are adjacent. Since the graph $\Gamma_0(\Delta)$ is connected, any two distinct vertices are adjacent. Let x_1, x_2, x_3 be three distinct vertices of Δ . It was established above that they are mutually adjacent. Then the edges x_1x_2 and x_2x_3 are adjacent and we get the claim. \square

The previous lemma can be generalized as follows.

Proposition 7. *If Δ is z -simple and there is a non-zero number $k < n - 1$ such that any two faces of the same non-zero rank $\leq k$ are z -connected, then Δ is $(k+2)$ -neighborly.*

Proof. The statement coincides with Lemma 6 if $k = 1$. Let $k \geq 2$. Lemma 6 states that Δ is 3-neighborly. It follows from Lemma 5 that for every $i \in [k]$ two i -faces are adjacent if their intersection is a $(i - 1)$ -face. Therefore, if X is a 4-element subset in the vertex set, then any two distinct 3-element subsets of X are adjacent 2-faces and X is a 3-face. Step by step, we establish that every subset consisting of not greater than $k + 2$ vertices is a face. \square

Proposition 7 together with Fact 1 give the following.

Corollary 4. *Suppose that, as in Proposition 7, Δ is z -simple and there is a non-zero number $k < n - 1$ such that any two faces of the same non-zero dimension $\leq k$ are z -connected. If $k > \lfloor n/2 \rfloor - 2$, then Δ is the n -simplex.*

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